1 Cohomology (on Sheaves)

For subsections 1.1 and 1.2 let K, K' and K'' be abelian categories. Furthermore let "Iff" be an abreviation of "If and only if".

1.1 Injective Objects

Lemma 1.1.1 (Exactness of $\operatorname{Hom}(-, E)$)**.** *The Functor* $\operatorname{Hom}(-, E)$ *is left exact, that is, if* $0 \to A \to B \to C \to 0$ *ist exact in* K*, then the derived sequence*

$$0 \to \operatorname{Hom}(A, E) \to \operatorname{Hom}(B, E) \to \operatorname{Hom}(C, E)$$

is exact in the Category of abelian groups.

Furthermore is $\operatorname{Hom}(-, E)$ *exact, if and only if* E *fullfills the following equivalent properties:*

(i) For all monomorphisms $A \hookrightarrow B$ *and morphisms* $A \xrightarrow{f} E$*, there exists an extension of* f *to a morphism* $B \to E$*, such that the following triangle commutes:*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ {\scriptstyle f}\downarrow & \swarrow & \\ E & & \end{array}$$

(ii) Every short exact sequence $0 \to E \to A \to B \to 0$ *splits, meaning it is isomorphic to*

$$0 \to E \xrightarrow{\iota_1} E \oplus B \xrightarrow{\pi_2} B \to 0.$$

Definition 1.1.2 (Injective Objects)**.** Iff $E \in \operatorname{Ob}(K)$ is such that $\operatorname{Hom}(-, E)$ is exact, we call E an *injective object (of K)*. Furthermore we say that K *has enough injectives*, iff for all $A \in \operatorname{Ob}(K)$ there is an injective E and a monomorphism $A \hookrightarrow E$.

Lemma 1.1.3 (Injective products and divisble groups)**.**

(i) Let $E_i \in \operatorname{Ob}(K)$*, then the product* $\prod_{i \in I} E_i$ *is injective, if and only if* E_i *is injective for all* $i \in I$*.*

(ii) An abelian group G *is injective (in Abgp), if and only if* G *is divisible, meaning that for all* $g \in G$ *and* $n \in \mathbb{Z}^\times$ *there exists an* $h \in G$ *with* $nh = g$*.*

Remark 1.1.4. The proof of Lemma 1.1.3 (ii) uses Zorn's Lemma, which itself is equivalent to the axiom of choice.

Lemma 1.1.5.

(i) For $F \in R\text{−}\operatorname{Mod}, G \in \operatorname{Abgp}$ *there exists a natural isomorphism of abelian groups*

$$\operatorname{Hom}_R(F, \operatorname{Hom}_{\mathbb{Z}}(R, G)) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}(F, G).$$

(ii) If G is an injective abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is an injective R-Module.

(iii) Suppose E is an injective R-module, such that for any R-module F we have:

$$\forall f \in F,\ f \neq 0 \Rightarrow \exists \phi \in \mathrm{Hom}_R(F, G) \text{ such that } \phi(f) \neq 0, \tag{1}$$

then any R-module can be embedded in an injective (namely a product of copies of E).

Proof.

(i) One can easily check this statement using the morphisms

$$\begin{aligned} \mathrm{Hom}_R(F, \mathrm{Hom}_{\mathbb{Z}}(R, G)) &\to \mathrm{Hom}_{\mathbb{Z}}(F, G) \qquad & \mathrm{Hom}_{\mathbb{Z}}(F, G) &\to \mathrm{Hom}_R(F, \mathrm{Hom}_{\mathbb{Z}}(R, G)) \\ \phi &\mapsto (f \mapsto \phi(f)(1)) & \psi &\mapsto (f \mapsto (r \mapsto \psi(rf))) \end{aligned}$$

(ii) Using statement (i) for $E = \mathrm{Hom}_{\mathbb{Z}}(R, G)$ we obtain a natural isomorphism $\mathrm{Hom}_R(-, E) \cong \mathrm{Hom}_{\mathbb{Z}}(-, G)$, where the right side is exact, since G is injective in Abgp.

(iii) Consider the map

$$\begin{aligned} \Phi \colon F &\to \prod_{\phi \in \mathrm{Hom}_R(F,E)} E \\ f &\mapsto (\phi(f))_{\phi \in \mathrm{Hom}_R(F,E)}. \end{aligned}$$

The right side is injective by Lemma 1.1.3 (i) and the R-module morphism Φ is monomorphic iff its kernel is zero. In other words: Φ is monomorphic iff for all $f \in F$ we get $f \neq 0 \Rightarrow f \notin \ker(\Phi)$.

□

Remark 1.1.6. Statement (i) is a special case of $\mathrm{Hom}_R(-, \mathrm{Hom}_S(-, -)) \cong \mathrm{Hom}_S(- \otimes_R -, -)$.

Theorem 1.1.7. *The category $R-$Mod has enough injectives for any Ring R.*

Proof. Use Lemma 1.5 (iii) to show the statement:

Consider the R-module $E = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, then by Lemma 1.1.3 (ii) (G injective iff divisible) and Lemma 1.1.5 (ii) ($\mathrm{Hom}_{\mathbb{Z}}(R, G)$ injective if G injective/divisble) E is injective.

We now want to check statement (1). So let $0 \neq f \in F \in \mathrm{Ob}(R-\mathrm{Mod})$. There exists a non-zero $\mathbb{Z}$-modulemorphism $\mathbb{Z} \cdot f \to \mathbb{Q}/\mathbb{Z}$ (using $f \mapsto 1/2 + \mathbb{Z}$ if f has infinite order or $f \mapsto 1/n + \mathbb{Z}$ if f has finite order n). Because $\mathbb{Q}/\mathbb{Z}$ is injective, this extends to a $\mathbb{Z}$-module morphism $\psi \colon F \to \mathbb{Q}/\mathbb{Z}$, such that $\psi(f) \neq 0$ (Lemma 1.1.1 (i)). Now Lemma 1.1.5 (i) gives corresponding $\phi \in \mathrm{Hom}_R(F, E)$ with $\phi(f) \neq 0$. Hence Lemma 1.1.5 (iii) applies. □

1.2 Derived functors

Definition 1.2.1 (Complexes and morphisms)**.** A *complex (right-cocomplex)* $L^\bullet$ in K is a family $(L^n, d_n\colon L^n \to L^{n+1})_{n\in\mathbb{N}}$ of objects and morphisms, such that for all $n \in \mathbb{N}$ we have $d_{n+1} \circ d_n = 0$:

$$L^0 \xrightarrow{d_0} L^1 \xrightarrow{d_1} \dots \xrightarrow{d_{i-1}} L^n \xrightarrow{d_n} L^{n+1} \xrightarrow{d_{n+1}} L^{n+2} \xrightarrow{d_{n+2}} \dots \qquad \left(L^n \xrightarrow[d_{n+1}\circ d_n]{0} L^{n+2}\right)$$

The *n-th cohomology of the complex* $L^\bullet$ is defined as

$$H^n(L^\bullet) := \ker(d_n)/\operatorname{im}(d_{n-1})$$

for all $n \in \mathbb{N}$, with $L^{-1} := 0$. The set $H^*(L^\bullet) := \{H^n(L^\bullet) \mid n \in \mathbb{N}\}$ is called *cohomology of* $L^\bullet$. Furthermore we call the complex $L^\bullet$ *exact (or acyclic)* iff for all $n \in \mathbb{N} \setminus \{0\}$ we have $H^n(L^\bullet) = 0$, that is, iff

$$0 \to H^0(L^\bullet) \to L^0 \to L^1 \to \dots$$

is exact. Let $A \in \mathrm{Ob}(K)$, then a *complex over* A is a complex $L^\bullet$ with $H^0(L^\bullet) \cong A$, and if $L^\bullet$ is exact, it is called a *resolution of* A. If furthermore L^n is injective for all $n \in \mathbb{N}$, we say that $L^\bullet$ is an *injective resolution of* A.

Additionally we define a morphism of complexes $g\colon L^\bullet \to M^\bullet$ to be a family of K-morphisms $g_n\colon L^n \to M^n$ for each $n \in \mathbb{N}$, such that the diagramm

$$\begin{array}{ccc} L^n & \xrightarrow{d_n} & L^{n+1} \\ {\scriptstyle g_n}\big\downarrow & & \big\downarrow{\scriptstyle g_{n+1}} \\ M^n & \xrightarrow{d'_n} & M^{n+1} \end{array}$$

commutes. Lastly we define two morphisms $g, h\colon L^\bullet \to M^\bullet$ to be (chain) homotopic (Notation: $g \simeq h$), iff there are K-morphisms $k_n\colon L^{n+1} \to M^n$ for all $n \in \mathbb{N}$, such that $d'_{n-1}k_{n-1} + k_n d_n = g_n - h_n$

$$\begin{array}{ccccc} & & L^n & \xrightarrow{d_n} & L^{n+1} \\ & {\scriptstyle k_{n-1}}\swarrow & {\scriptstyle g}\big\downarrow\big\downarrow{\scriptstyle h} & \swarrow{\scriptstyle k_n} & \\ M^{n-1} & \xrightarrow{d'_{n-1}} & M^n & & \end{array}$$

Note that g and h are already homotopic, if the diagramms

$$\begin{array}{ccc} & & L^n \\ & {\scriptstyle k_{n-1}}\swarrow & \big\downarrow{\scriptstyle g} \\ M^{n-1} & \xrightarrow{d'_{n-1}} & M^n \end{array} \qquad\qquad \begin{array}{ccc} L^n & \xrightarrow{d_n} & L^{n+1} \\ \big\downarrow{\scriptstyle h} & \swarrow{\scriptstyle k_n} & \\ M^n & & \end{array}$$

both commute. However this is *not* a necessary condition.

Lemma 1.2.2. *A morphism of complexes $g\colon L^\bullet \to M^\bullet$ induces morphisms of cohomology $H(g) = g^*\colon H^*(L^\bullet) \to H^*(M^\bullet)$ in a functorial way.*

Lemma 1.2.3. *If $L^\bullet$ (resp. $M^\bullet$) is an injective resolution of A (resp. B), then any morphism $f\colon A \to B$ can be lifted to a morphism $g\colon L^\bullet \to M^\bullet$, meaning that $f = g^*\colon H^0(L^\bullet) \to H^0(M^\bullet)$. This morphism is unique up to isomorphism.*

Corollary 1.2.4. *If K has enough injectives, then every object $A \in \mathrm{Ob}(K)$ has an injective resolution and any two injective resolutions $L^\bullet$ and $M^\bullet$ of A are homotopy equivalent. That is, there are morphisms $g\colon L^\bullet \to M^\bullet$ and $h\colon M^\bullet \to L^\bullet$, such that $g \circ h \simeq \mathrm{id}_M$ and $h \circ g \simeq \mathrm{id}_L$.*

Definition 1.2.5 (Right derived functors)**.** Let $F\colon K \to K'$ be a left exact functor, where K, K' are abelian categories and K has enough injectives. Given $A \in \mathrm{Ob}(K)$, let $L^\bullet$ be an injective resoulution of A, and let $(R^nF)(A) = H^n(FK^\bullet)$ be the cohomology of the complex $(F(K^n), F(d_n))_{n\in\mathbb{N}}$, then $R^nF\colon K \to K'$ are called *right derived functors of F*.

In order to define R^nF on a map $f\colon A \to B$ in K, lift f to a morphism of complexes $g\colon L^\bullet \to M^\bullet$, where $L^\bullet, M^\bullet$ are injective resoltions of A and B respectively and obtain a map

$$R^nf\colon R^nF(A) = H^n(FL^\bullet) \xrightarrow{F(g)^*} H^n(FM^\bullet) = R^nF(B).$$

Lemma 1.2.6.

(a) The R^nF are well defined, meaning that they are independent of the choices of injective resolutions and of maps between them.

(b) The Functor R^0F is naturally isomorphic to F.

(c) If A is injective we get $R^nF(A) = 0$ for all $n > 0$.

Definition 1.2.7 (∂-functor)**.** Let $a \in \mathbb{N} \cup \{\infty\}$. A ∂-functor $T^\bullet\colon K \to K'$ is a sequence of functors $\{T^n\colon K \to K'; 0 \le n < a\}$ together with an assignment to each short exact sequence

$$0 \to A \to B \to C \to 0$$

in K of a collection of morphisms $\partial = \partial_T\colon T^{n-1}C \to T^nA$ $(0 < n < a)$, such that

(i) If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

commutes in K and has exact rows, then the corresponding diagrams

$$\begin{array}{ccc} T^{n-1}C & \xrightarrow{\partial} & T^nA \\ {\scriptstyle T^{n-1}g}\downarrow & & \downarrow{\scriptstyle T^n f} \\ T^{n-1}C' & \xrightarrow{\partial} & T^nA' \end{array}$$

commute, meaning ∂ is natural.

(ii) If $0 \to A \to B \to C \to 0$ is exact, the asociated sequence

$$0 \to T^0A \to T^0B \to T^0C \xrightarrow{\partial} T^1A \to T^1B \to T^1C \xrightarrow{\partial} \cdots \to T^{n-1}C \xrightarrow{\partial} T^nA \to \dots$$

for $n < a$ is a complex. The ∂-functor is called *exact* iff for any sequence the corresponding sequence is always exact.

A *morphism* of ∂-functors (with the same a) $S^\bullet \to T^\bullet$ is given by a sequence of natural transformations $\{S^n \to T^n \mid 0 \leq n < a\}$, such that for any short exact sequence $0 \to A \to B \to C \to 0$ the diagrams

$$\begin{array}{ccc} S^{n-1}C & \xrightarrow{\partial_S} & S^nA \\ \downarrow & & \downarrow \\ T^{n-1}C & \xrightarrow{\partial_T} & T^nA \end{array}$$

commute.

If $F\colon K \to K'$ is a functor, a *∂-functor over* F is a ∂-functor $\{T^n, \partial_T\}$ together with a natural isomorphism $F \xrightarrow{\sim} T^0$, hence if T is exact, F is left exact.

Lemma 1.2.8. *If $F\colon K \to K'$ is a left-exact functor between abelian categories and $0 \to A \to B \to C \to 0$ is a split exact sequence in K, then*

$$0 \to FA \to FB \to FC \to 0$$

is also split exact (hence $F(A \oplus C) \cong FA \oplus FC$).

Proof. The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff $A \to B \to C$ is exact and there exists a diagramm $A \xleftarrow{f'} B \xleftarrow{g'} C$, such that $f' \circ f = \mathrm{id}_A$ and $g \circ g' = \mathrm{id}_C$. Applying the functor F to all of these objects and morphisms does not change these properties. □

Lemma 1.2.9. *Suppose $L^\bullet \to M^\bullet \to N^\bullet$ are morphisms of complexes in an abelian category K, such that for all $n \in \mathbb{N}$ the sequence*

$$0 \to L^n \to M^n \to N^n \to 0$$

is exact (we call this a short exact sequence of complexes*), then there is a collection of morphisms $\partial\colon H^n(N^\bullet) \to H^{n+1}(L^\bullet)$ for all n, such that the sequence*

$$0 \to H^0(L)^\bullet \to H^0(M^\bullet) \to H^0(N^\bullet) \xrightarrow{\partial} H^1(L^\bullet) \to \dots$$

is exact. Moreover ∂ is natural in the sense, that if

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & N^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1^\bullet & \longrightarrow & M_1^\bullet & \longrightarrow & N_1^\bullet & \longrightarrow & 0
\end{array}$$

is a commutative diagram of morphisms of complexes, with exact rows, then the induced morphism between the long exact sequences is a morphism of complexes, that is for all $n \in \mathbb{N}$ the diagram

$$\begin{array}{ccc}
H^n(N^\bullet) & \xrightarrow{\partial} & H^{n+1}(L^\bullet) \\
\downarrow & & \downarrow \\
H^n(N_1^\bullet) & \xrightarrow{\partial_1} & H^{n+1}(L_1^\bullet)
\end{array}$$

commutes (where ∂_1 is constructed from the lower exact row).

Theorem 1.2.10. *If $F\colon K \to K'$ is a left-exact functor between abelian categories, where K has enough injectives, then the sequence of functors $R^\bullet F = \{R^n F \mid n \in \mathbb{N}\}$ forms an exact ∂-functor over F.*

Proof. We need to show that there is a natural assignment to each short exact sequence $0 \to A \to B \to C \to 0$ in K of a long exact sequence

$$0 \to FA \to FB \to FC \longrightarrow R^1FA \to \cdots \to R^{n-1}FC \longrightarrow R^nFA \to R^nFB \to R^nFC \to \dots \quad (2)$$

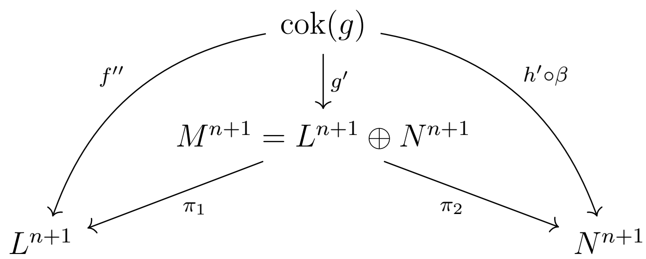
Let $L^\bullet$ (resp. $N^\bullet$) be an injective resolution of A (resp. C). We first construct a complex of injective objects $M^\bullet$ over B and a short exact sequence of complexes $0 \to L^\bullet \to M^\bullet \to N^\bullet \to 0$. Assume this has been done up to the situation in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{n-1} & \longrightarrow & M^{n-1} & \longrightarrow & N^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^n & \longrightarrow & M^n & \longrightarrow & N^n & \longrightarrow & 0
\end{array}$$

for $n \in \mathbb{N}$ (with convention $L^{-1} = A, M^{-1} = B, N^{-1} = C$), then we can construct the diagramm

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{n-1} & \longrightarrow & M^{n-1} & \longrightarrow & N^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & L^n & \longrightarrow & M^n & \longrightarrow & N^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{cok}(f) & \xrightarrow{\alpha} & \operatorname{cok}(g) & \xrightarrow{\beta} & \operatorname{cok}(h) & & \\
 & & \downarrow{\scriptstyle f'} & & & & \downarrow{\scriptstyle h'} & & \\
0 & \longrightarrow & L^{n+1} & \longrightarrow & M^{n+1} := L^{n+1} \oplus N^{n+1} & \longrightarrow & N^{n+1} & \xrightarrow{h} & 0,
\end{array}$$

where f', h' arise from UVP of cokernels. One can show that the third row is exact ([Mac1] VIII par. 4). M^{n+1} is injective because of Lemma 1.1.3 (prod. of injective objects). Define $g' : \text{cok}(g) \to M^{n+1}$ by



with f'' being an extension of f' to $\text{cok}(g)$ as in Lemma 1.1.1 (i), since L^{n+1} is injective, and α is monomorphic. One can show that $M^\bullet$ is a complex, and that $0 \to L^\bullet \to M^\bullet \to N^\bullet \to 0$ is a short exact sequence.

The sequence (2) collapses (since $L^\bullet$ and $N^\bullet$ are exact) into a collection of exact pieces $0 \to H^n(M^\bullet) \to 0$ for $n \in \mathbb{N}$, which shows that $H^n(M^\bullet) = 0$ for all $n \geq 1$. Hence $M^\bullet$ is exact and therefore an injective resolution of B. Therefore one may use $M^\bullet$ to compute $R^nF(B)$.

Since the short exact sequences $0 \to L^n \to M^n \to N^n \to 0$ is split exact by Lemma 1.2.8 (left-exact functor keeps split exactness) the sequence $0 \to FL^\bullet \to FM^\bullet \to FN^\bullet \to 0$ is exact and so Lemma 1.2.9 yields

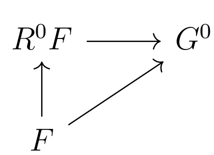
$$\cdots \to H^n(FL^\bullet) \to H^n(FM^\bullet) \to H^n(FN^\bullet) \longrightarrow H^{n+1}(FL^\bullet) \to \dots$$

which is the required sequence by Definition 1.2.5 and Lemma 1.2.6. The naturality of ∂ follows from Lemma 1.2.9. □

Theorem 1.2.11. *Let $F : K \to K'$ be a left exact functor between abelian categories, where K has enough injectives. Then the ∂-functor $R^\bullet F$ has the following universal property. Suppose that $\{G^n \mid 0 \leq n < a\}$ is a ∂-functor over F, then there is a unique morphism of ∂-functors*

$$\{R^nF \mid 0 \leq n < a\} \to \{G^n \mid 0 \leq n < a\}$$

such that the triangle



commutes. If furthermore $G^\bullet$ is exact and effaceable (meaning for any injective object E of K, we have $G^nE = 0$ for all $0 < n < a$), then the morphism $R \to G^\bullet$ is an isomorphism of ∂-functors over F.

Proof. For an object A of K we construct the morphisms $R^nFA \to G^nA$ by induction on n. For $n = 0$ we compose the isomorphisms which show that $R^\bullet F$ and $G^\bullet$ are ∂-functors over F. Embed A in an injective object E of K to obtain a short exact sequence

$$0 \to A \to E \to B \to 0.$$

Hyptohesis, Lemma 1.2.6, and the long sequences of this exact sequence, give us a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^{n-1}FB & \xrightarrow{f} & R^nFA & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g & & \\ G^{n-1}E & \longrightarrow & G^{n-1}B & \xrightarrow{h} & G^nA & \longrightarrow & G^nE \end{array}$$

with an exact top row. Hence f is an isomorphism, and so there is a unique map g making the diagram commute. This is independent of E and defines a morphism of ∂-functors. If $G^\bullet$ is exact and effaceable, the same diagram shows, by induction on n, that g is an isomorphism (since h is). □

Corollary 1.2.12. *Let $K \xrightarrow{F} K' \xrightarrow{G} K''$ be functors between abelian categories, where K and K' each has enough injectives. Suppose that*

(i) G is left-exact and

(ii) F is exact and transforms injectives in K into G-acyclic objects (meaning for all injective E in K we get $R^nG(FE) = 0$ for all $n > 0$,

then there is a natural isomorphism of ∂-functors

$$R^\bullet(G \circ F) \cong (R^\bullet G) \circ F.$$

The same holds, if instead F is left-exact and G is exact.